Polynomial decay rate for the dissipative wave equation

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1 Introduction and main result

This paper is devoted to study the stabilization of the linear wave equation in a bounded domain damped in a subdomain when the geometrical control condition (see [BLR]) of the work of C. Bardos, G. Lebeau and J. Rauch is not fulfilled. In such case, they [BLR] proved that the uniform exponential decay rate of the energy cannot be hoped due to the existence of a trapped ray that never reaches the support of the damping. Another important contribution in this field was done by G. Lebeau [Le] who establishes a logarithmic decay rate for the dissipative wave equation when no assumption on rays of geometrical optics is required, but when more regularity on the initial data is allowed. Now, it seems natural to search a general description of the geometries of both domain and support of the damping under which the energy of the dissipative wave equation decreases in a polynomial way. A first answer in this direction was done by Z. Liu and B. Rao [LR] who consider the wave equation on a square damped in a vertical strip. In this paper, we improve the geometry to a partially cubic domain where the damping acts in a neighborhood of the boundary except between a pair of parallel square face of the cube.

Before stating the main result of this paper, we begin by presenting precisely the geometry of our problem. Next, we introduce the equations that will be used throughout this work. Our main result is given at the end of this section.

1.1 The geometry

Let $m_1, m_2, \rho > 0$. Let Ω be a connected domain in $\mathbb{R}^3$ bounded by Γ_1, Γ_2, Υ where

$\Gamma_1 = [-m_1, m_1] \times [-m_2, m_2] \times \{\rho\}$, with boundary $\partial\Gamma_1$,

$\Gamma_2 = [-m_1, m_1] \times [-m_2, m_2] \times \{-\rho\}$, with boundary $\partial\Gamma_2$,

Υ is a surface such that $\Upsilon \subset \mathbb{R}^2 \setminus ((-m_1, m_1) \times (-m_2, m_2)) \times \mathbb{R}$, with boundary $\partial\Upsilon = \partial\Gamma_1 \cup \partial\Gamma_2$.

We suppose that the boundary of Ω, $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Upsilon$, is C^∞. Let Θ be a neighborhood of Υ in $\mathbb{R}^3$ and let $\omega = \Omega \cap \Theta$.

As Θ is a neighborhood of Υ in $\mathbb{R}^3$, there exists $r_o \in (0, \min(m_1, m_2, \rho)/2)$ such that $[x_{b1} - 2r_o, x_{b1} + 2r_o] \times [x_{b2} - 2r_o, x_{b2} + 2r_o] \times [x_{b3} - 2r_o, x_{b3} + 2r_o] \subset \Theta$ for any $(x_{b1}, x_{b2}, x_{b3}) \in \partial\Gamma_1$. Let $h_o = \min\left(1, (r_o/8)^2\right)$. Next, we choose $\omega_o = (-m_1 + r_o, m_1 - r_o) \times (-m_2 + r_o, m_2 - r_o) \times \left(-\frac{\rho}{4}, \frac{\rho}{4}\right)$.

Throughout this paper, c denotes a positive constant which only may depend on (m_1, m_2, ρ). Also γ will denote an absolute constant larger than one. The value of $c > 0$ and $\gamma > 1$ may change from line to line.

1.2 The equations

We consider the dissipative wave equation in Ω with initial data $(w_0, w_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$.

$$\left\{ \begin{array}{rll} \partial_t^2 w - \Delta w + \alpha(x) \partial_t w & = 0 & \text{in } \Omega \times \mathbb{R}_+ \\ w & = 0 & \text{on } \partial\Omega \times \mathbb{R}_+ \\ (w(\cdot,0), \partial_t w(\cdot,0)) & = (w_0, w_1) & \text{in } \Omega \ , \end{array} \right. \tag{1.1}$$

with a non-negative dissipative potential $\alpha \in L^\infty(\Omega)$ such that $\alpha > 0$ in ω. Denote

$$\mathcal{E}(w,t) = \int_\Omega \left(|\partial_t w(x,t)|^2 + |\nabla w(x,t)|^2 \right) dx = \mathcal{E}(w,0) - 2\int_0^t \int_\Omega \alpha(x) |\partial_t w(x,\theta)|^2 \, dx d\theta \ ,$$

and recall that $\mathcal{E}(w,t)$ is a continuous decreasing function of time.

Denote by $\{\mu_j\}_{j\geq 1}$, $0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots$, the eigenvalues of $-\Delta$ on Ω with Dirichlet boundary conditions and by $\{\ell_j\}_{j\geq 1}$ the corresponding normalized eigenfunctions, i.e., $\|\ell_j\|_{L^2(\Omega)} = 1$. Let $u = u(x,t)$ be the solution of the wave equation

$$\left\{ \begin{array}{rll} \partial_t^2 u - \Delta u & = 0 & \text{in } \Omega \times \mathbb{R} \\ u & = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ (u(\cdot,0), \partial_t u(\cdot,0)) & = (u_0, u_1) & \text{in } \Omega \ . \end{array} \right. \tag{1.2}$$

Suppose that $u_0 = \sum_{j\geq 1} b_j^0 \ell_j$ and $u_1 = \sum_{j\geq 1} b_j^1 \ell_j$ are such that

$$\|u_0\|^2_{H^2 \cap H_0^1(\Omega)} = \sum_{j\geq 1} \mu_j^2 \left| b_j^0 \right|^2 < +\infty \text{ and } \|u_1\|^2_{H_0^1(\Omega)} = \sum_{j\geq 1} \mu_j \left| b_j^1 \right|^2 < +\infty \ ,$$

then it is known that

$$u(\cdot,t) = \sum_{j\geq 1} \left[b_j^0 \cos\left(t\sqrt{\mu_j}\right) + \frac{b_j^1}{\sqrt{\mu_j}} \sin\left(t\sqrt{\mu_j}\right) \right] \ \ell_j \ ,$$

and $u \in C\left(\mathbb{R}; H^2(\Omega) \cap H_0^1(\Omega)\right) \cap C^1\left(\mathbb{R}; H_0^1(\Omega)\right) \cap C^2\left(\mathbb{R}; L^2(\Omega)\right)$. Let

$$\mathcal{G}(u,t) = \int_\Omega \left(|\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2 \right) dx = \mathcal{G}(u,0) \ .$$

1.3 The bicharacteristics

It is usual to associate with the wave equation, the geodesics. Recall that the bicharacteristics are curves in the space-time variables and their Fourier variables described by

$$\left\{ \begin{array}{rl} x(s) & = x_o + 2\xi(s) s \\ t(s) & = t_o - 2\tau(s) s \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rl} \xi(s) & = \xi_o \\ \tau(s) & = \tau_o \end{array} \right.$$

with $|\xi(s)|^2 - \tau^2(s) = 0$ for $s \in [0, +\infty)$, when $(x_o, t_o, \xi_o, \tau_o) \in \mathbb{R}^4 \times \mathbb{R}^4 \setminus \{0\}$. The rays are the projection of the bicharacteristics on the space-time domain

$$\left\{ \begin{array}{rl} x(s) - x_o - 2\xi_o s & = 0 \\ t(s) + 2\tau_o s & = 0 \\ |\xi_o|^2 - \tau_o^2 & = 0 \ , \end{array} \right.$$

here, $t_o = 0$ and $\tau_o \neq 0$. The generalized rays are rays taking into account the geometry by following the rules of optic geometric.

The key geometric observation in our setting is that we do not have that any generalized ray meets ω. More precisely, any $(x_o, t_o, \xi_o, \tau_o) \in \omega_o \times \mathbb{R} \times \mathbb{R}^4 \backslash \{0\}$ such that $\xi_o = (0,0,\pm 1)$ generates a trapped generalized ray which never goes outside $[-m_1, m_1] \times [-m_2, m_2] \times [-\rho, \rho]$. As a result, we do not have an uniform exponential decay for the dissipative wave equation for any damping only acting in ω. Of course, the logarithmic decay rate still holds from Carleman inequalities. However, we may hope a better decay rate because our trapped generalized ray behaves quite simply by bouncing between Γ_1 and Γ_2 always in the same direction $\xi_o = (0,0,\pm 1)$. Furthermore, observe that the geometrical control condition is fulfilled with $\omega \cup \omega_o$.

1.4 Main result

Now we are ready to formulate our main result.

Theorem.- *There exist $C > 0$ and $\delta > 0$ such that for any $t > 0$ and any initial data $(w_0, w_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, the solution w of (1.1) satisfies,*

$$\int_\Omega \left(\left|\partial_t w(x,t)\right|^2 + \left|\nabla w(x,t)\right|^2 \right) dx \leq \frac{C}{t^\delta} \left\| (w_0, w_1) \right\|^2_{H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)} \ . \tag{1.3}$$

Our strategy to get such polynomial decay rate will consist to establish an observability estimate for the wave equation u solution of (1.2), or more precisely, an inequality which traduces the unique continuation property for u between $\omega \times (0,T)$ and $\Omega \times \{0\}$ for some $T < +\infty$. For example, it is now known that an interpolation inequality of the form

$$\left\| (u_0, u_1) \right\|^2_{H_0^1(\Omega) \times L^2(\Omega)} \leq C \ \Lambda^{1/\delta} \int_0^T \int_\omega \left|\partial_t u(x,t)\right|^2 dxdt$$

where $\Lambda = \frac{\left\| (u_0, u_1) \right\|^2_{H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)}}{\left\| (u_0, u_1) \right\|^2_{H_0^1(\Omega) \times L^2(\Omega)}}$, implies (1.3). On another hand, it is not difficult to deduce from (1.3) an inequality in the following form

$$\left\| (u_0, u_1) \right\|^2_{H_0^1(\Omega) \times L^2(\Omega)} \leq c \int_0^{(C\Lambda)^{1/\delta}} \int_\Omega \alpha(x) \left|\partial_t u(x,t)\right|^2 dxdt$$

and conversely, the later inequality implies (1.3) where the constant C may change of values. In this paper, (1.3) comes from

$$\left\| (u_0, u_1) \right\|^2_{H_0^1(\Omega) \times L^2(\Omega)} \leq C \ \Lambda^\gamma \int_0^{(C\Lambda)^\gamma} \int_\omega \left|\partial_t u(x,t)\right|^2 dxdt \ .$$

As it was suggested in the previous subsection, we have to pay more attention on a ray of geometrical optics bouncing up and down infinitely between the two parallel planes Γ_1 and Γ_2. In same time, we need to estimate in a good way the dissipation phenomena in order to improve the logarithmic decay rate. To this end, we apply some simple tools usually used in the propagation of singularities [AG]-[CV]-[R] in order to link the number of reflections and the s variable of the bicharacteristic flow. In particular, we will work with the operator $O_p(D) = i\partial_s + h\left(\Delta - \partial_t^2\right)$ for $h \in (0, h_o]$. Observe that the product of four, mono-dimensional, solutions of the Schrödinger equation $i\partial_s \pm h\partial^2$ can create a solution of $O_p(D)\, a(x,t,s) = 0$. The dispersive property of the linear Schrödinger equation $i\partial_s \pm h\partial^2$ will be exploited.

The next section describes an interpolation inequality. In section 3, we give the proof of Theorem. We close this paper with two Appendixes devoted to prove some technical results.

2 Interpolation inequality

The purpose of this section is to establish the following inequality.

Theorem 2 .- *Let $T>0$. There exist $C>0$ and $\gamma>1$ such that for any $h\in(0,h_o]$ and initial data $(u_0,u_1)\in H^2(\Omega)\cap H_0^1(\Omega)\times H_0^1(\Omega)$, the solution u of (1.2) satisfies*

$$\begin{array}{ll}\int_{\omega_o}\int_0^T\left|\partial_t u(x,t)\right|^2 dxdt & \leq C\left(\frac{1}{h}\right)^{\gamma}\left(\int_\Upsilon\int_{|t|\leq\gamma\left(\frac{1}{h}\right)^{\gamma}}\left|\partial_\nu u(x,t)\right|^2 dxdt\right)^{1/2}\left\|(u_0,u_1)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}\\ & +C\sqrt{h}\ \left\|(u_0,u_1)\right\|_{H^2\cap H_0^1(\Omega)\times H_0^1(\Omega)}\left\|(u_0,u_1)\right\|_{H_0^1(\Omega)\times L_0^2(\Omega)}\ .\end{array}$$

The rest of this section is devoted to the proof of Theorem 2. We begin to introduce some weight functions inspired by [Z], with the properties of localization, propagation and dispersion. Next, we make appear the Fourier variables and introduce some Fourier integral operators dependent on the number of reflections between Γ_1 and Γ_2. Their properties at the boundary are analyzed. The process of propagation is then applied. Some parameters are adequately chosen and that will complete the proof.

2.1 The weight functions

The hypothesis saying that $\Upsilon\subset\mathbb{R}^2\backslash((-m_1,m_1)\times(-m_2,m_2))\times\mathbb{R}$ implies that for any $x_o\in\overline{\omega_o}$, $B(x_o,r_o/2)\cap\partial\Omega=\emptyset$, where $B(x_o,r)$ denotes the ball of center x_o and radius r. We introduce $\chi_{x_o}=\chi\in C_0^\infty(B(x_o,r_o/2))$ be such that $0\leq\chi\leq1$ and $\chi=1$ on $B(x_o,r_o/4)$.

From any point $x_o\in\overline{\omega_o}$, we will localize around x_o and eventually be able to propagate some local regularity. To this end, let $h\in(0,h_o]$ and let us define

$$a(x,t,s)=\left(\frac{1}{(is+1)^{3/2}}\ e^{-\frac{1}{4h}\frac{x^2}{is+1}}\right)\left(\frac{1}{\sqrt{-ihs+1}}\ e^{-\frac{1}{4}\frac{t^2}{-ihs+1}}\right)\ ,$$

$$a_o(x,t)=a(x-x_o,t,0)\ \text{ and }\ \varphi(x,t)=\chi(x)\,a_o(x,t)\ .$$

We get the following identities

$$\left(i\partial_s+h\left(\Delta-\partial_t^2\right)\right)a(x,t,s)=0\quad\forall(x,t,s)\in\Omega\times\mathbb{R}\times(0,+\infty)\ ,$$

$$|a(x,t,s)|=\frac{1}{\left(\sqrt{s^2+1}\right)^{3/2}}\frac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}\ e^{-\frac{x^2}{4h}\frac{1}{s^2+1}}\ e^{-\frac{t^2}{4}\frac{1}{(hs)^2+1}}\ . \tag{2.1}$$

Now we use such weight functions a_o and φ with u the solution of the wave equation (1.2) as follows. By integrations by parts,

$$\begin{array}{l}\int_{\Omega\times\mathbb{R}}\chi(x)\left|a_o\partial_t u(x,t)\right|^2 dxdt\\ =-\int_{\Omega\times\mathbb{R}}\chi(x)\,\partial_t\left(\left|a_o(x,t)\right|^2\right)\frac{1}{2}\partial_t\left(\left|u(x,t)\right|^2\right)dxdt-\int_{\Omega\times\mathbb{R}}\chi(x)\left|a_o(x,t)\right|^2\partial_t^2u(x,t)\,u(x,t)\,dxdt\\ =\int_{\Omega\times\mathbb{R}}\chi(x)\,\frac{1}{2}\partial_t^2\left(\left|a_o(x,t)\right|^2\right)\left|u(x,t)\right|^2dxdt-\int_{\Omega\times\mathbb{R}}\chi(x)\left|a_o(x,t)\right|^2\partial_t^2u(x,t)\,u(x,t)\,dxdt\ .\end{array}$$

As $\partial_t^2\left(\left|a_o(x,t)\right|^2\right)=-\left|a_o(x,t)\right|^2+t^2\left|a_o(x,t)\right|^2$ and $t^2\left|a_o(x,t)\right|^2\leq4\left|a_o\left(x,t/\sqrt{2}\right)\right|^2$, we have

$$\begin{array}{l}\int_{\Omega\times\mathbb{R}}\chi(x)\left|a_o\partial_t u(x,t)\right|^2 dxdt\\ \leq2\int_{\Omega\times\mathbb{R}}a_o\left(x,t/\sqrt{2}\right)\varphi\left(x,t/\sqrt{2}\right)\left|u(x,t)\right|^2dxdt+\left|\int_{\Omega\times\mathbb{R}}a_o(x,t)\,\varphi(x,t)\,\partial_t^2u(x,t)\,u(x,t)\,dxdt\right|\ .\end{array} \tag{2.2}$$

Let $\{G_i; i \in I\}$ be a family of open sets covering ω_o, i.e., $\overline{\omega_o} \subset \bigcup_{i\in I} G_i$, such that $G_i = \left\{\left|x - x_o^i\right| \leq 2\sqrt{h}\right\}$ where $\left\{x_o^i\right\}_{i\in I} \in \overline{\omega_o}$ and I is a countable set such that the number of elements of I is bounded by $\frac{c_o}{h\sqrt{h}}$ for some constant $c_o > 0$ independent of $h \in (0, h_o]$. Consequently,

$$\begin{array}{ll} \int_{\omega_o\times(0,T)} \left|\partial_t u(x,t)\right|^2 dxdt & \leq e^{\frac{1}{2}T^2} \int_{\omega_o\times(0,T)} e^{-\frac{1}{2}t^2} \left|\partial_t u(x,t)\right|^2 dxdt \\ & \leq e^{\frac{1}{2}T^2+2} \sum_{i\in I} \int_{G_i\times\mathbb{R}} \chi_{x_o^i}(x) \left|a\left(x - x_o^i, t, 0\right) \partial_t u(x,t)\right|^2 dxdt \\ & \leq e^{\frac{1}{2}T^2+2} \sum_{i\in I} \int_{\Omega\times\mathbb{R}} \chi_{x_o^i}(x) \left|a\left(x - x_o^i, t, 0\right) \partial_t u(x,t)\right|^2 dxdt \end{array} \tag{2.3}$$

and we will search to bound $\int_{\Omega\times\mathbb{R}} \chi_{x_o^i}(x) \left|a\left(x - x_o^i, t, 0\right) \partial_t u(x,t)\right|^2 dxdt$ by a suitable term $E_h(u)$ independent of i in order to get

$$\int_{\omega_o\times(0,T)} \left|\partial_t u(x,t)\right|^2 dxdt \leq \frac{C_o E_h(u)}{h\sqrt{h}}$$

for some $C_o > 0$ independent of u and h. To this end, we will first study in the next subsections the second term of the second member of (2.2),

$$\int_{\Omega\times\mathbb{R}} a_o(x,t)\,\varphi(x,t)\, f(x,t)\, u(x,t)\, dxdt \text{ when } f = \partial_t^2 u.$$

2.2 The Fourier variables

Denote

$$\widehat{\varphi f}(\xi,\tau) = \int_{\Omega\times\mathbb{R}} e^{-i(x\xi+t\tau)}\ \varphi(x,t)\, f(x,t)\ dxdt\ ,$$

then for any $(x,t) \in \Omega\times\mathbb{R}$,

$$a_o(x,t)\,\varphi(x,t)\, f(x,t) = a_o(x,t)\, \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{i(x\xi+t\tau)}\ \widehat{\varphi f}(\xi,\tau)\ d\xi d\tau\ .$$

Let $\lambda \geq 1$. We cut the integral over $\tau \in \mathbb{R}$ into two parts, $\{|\tau| \geq \lambda\}$ and $\{|\tau| < \lambda\}$. Next, for $\{|\tau| < \lambda\}$ and $\xi = (\xi_1, \xi_2, \xi_3)$, the integral over $\xi_3 \in \mathbb{R}$ is divided as follows. Denote $(2\mathbb{Z}+1) = \{2n+1 \backslash n \in \mathbb{Z}\}$, then

$$\begin{array}{ll} a_o(x,t)\,\varphi(x,t)\, f(x,t) & = a_o(x,t)\, \frac{1}{(2\pi)^4} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} e^{i(x\xi+t\tau)}\ \widehat{\varphi f}(\xi,\tau)\ d\xi d\tau \\ & \quad + a_o(x,t)\, \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau|\geq\lambda} e^{i(x\xi+t\tau)}\ \widehat{\varphi f}(\xi,\tau)\ d\xi d\tau\ . \end{array}$$

Consequently,

$$\begin{array}{l} \int_{\Omega\times\mathbb{R}} a_o(x,t)\,\varphi(x,t)\, f(x,t)\, u(x,t)\, dxdt - R_0 \\ = \int_{\Omega\times\mathbb{R}} a_o(x,t)\, \frac{1}{(2\pi)^4} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} e^{i(x\xi+t\tau)}\ \widehat{\varphi f}(\xi,\tau)\ d\xi d\tau\ u(x,t)\, dxdt \end{array} \tag{2.4}$$

where

$$R_0 = \int_{\Omega\times\mathbb{R}} a_o(x,t)\, \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau|\geq\lambda} e^{i(x\xi+t\tau)}\ \widehat{\varphi f}(\xi,\tau)\ d\xi d\tau\ u(x,t)\ dxdt\ .$$

Moreover, it holds

$$R_0 \leq c\,\sqrt{\frac{1}{\lambda}} \sqrt{\mathcal{G}(u,0)} \sqrt{\mathcal{G}(\partial_t u,0)}\ . \tag{2.5}$$

The proof of (2.5) is given in Appendix A.

2.3 The Fourier integral operators

Denote $x=(x_1,x_2,x_3)\in\Omega$, $\xi=(\xi_1,\xi_2,\xi_3)$ with $(\xi_1,\xi_2)\in\mathbb{R}^2$. Let $(x_o,\xi_{o3})=(x_{o1},x_{o2},x_{o3},\xi_{o3})\in\overline{\omega_o}\times(2\mathbb{Z}+1)$, we introduce for all $s\geq 0$ and $n\in\mathbb{Z}$

$$\begin{array}{l}A_s^n\left(x_o,\xi_{o3}\right)f\left(x,t\right)\\ =\frac{(-1)^n}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}e^{i(x_1\xi_1+x_2\xi_2+t\tau)}\ e^{i\left[(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho\right]\xi_3}\ e^{-i\left(\xi^2-\tau^2\right)hs}\ \widehat{\varphi f}\left(\xi,\tau\right)\\ \quad a\left(x_1-x_{o1}-2\xi_1 hs,x_2-x_{o2}-2\xi_2 hs,x_3-(-1)^n\left[-2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho+x_{o3}+2\xi_3 hs\right],t+2\tau hs,s\right)d\xi d\tau\ .\end{array}$$

Let $(P,Q)\in\mathbb{N}^2$. Consider the solution

$$A_{s,P,Q}\left(x_o,\xi_{o3}\right)f\left(x,t\right)=\sum_{n=-2Q}^{2P+1}A_s^n\left(x_o,\xi_{o3}\right)f\left(x,t\right)\ .$$

One can check that for any $(x_o,\xi_{o3},P,Q)\in\overline{\omega_o}\times(2\mathbb{Z}+1)\times\mathbb{N}^2$,

$$\left(i\partial_s+h\left(\Delta-\partial_t^2\right)\right)A_{s,P,Q}\left(x_o,\xi_{o3}\right)f\left(x,t\right)=0\quad\forall\left(x,t,s\right)\in\Omega\times\mathbb{R}\times(0,+\infty)\ .\tag{2.6}$$

2.4 At $s=0$

Since

$$A_0^0\left(x_o,\xi_{o3}\right)f\left(x,t\right)=a_o\left(x,t\right)\frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}e^{i(x\xi+t\tau)}\ \widehat{\varphi f}\left(\xi,\tau\right)\ d\xi d\tau\ ,$$

we then obtain from (2.4) that

$$\int_{\Omega\times\mathbb{R}}a_o\left(x,t\right)\varphi\left(x,t\right)f\left(x,t\right)u\left(x,t\right)dxdt-R_0=\int_{\Omega\times\mathbb{R}}\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}A_0^0\left(x_o,\xi_{o3}\right)f\left(x,t\right)\ u\left(x,t\right)dxdt\ .$$

Observe that

$$A_{0,P,Q}\left(x_o,\xi_{o3}\right)f\left(x,t\right)=A_0^0\left(x_o,\xi_{o3}\right)f\left(x,t\right)+\sum_{n=1}^{2P+1}A_0^n\left(x_o,\xi_{o3}\right)f\left(x,t\right)+\sum_{-2Q\leq n\leq -1}A_0^n\left(x_o,\xi_{o3}\right)f\left(x,t\right)\ ,$$

with the convention that for $Q=0$, $\sum_{-2Q\leq n\leq -1}A_0^n\left(x_o,\xi_{o3}\right)f\left(x,t\right)=0$. So, we deduce that

$$\int_{\Omega\times\mathbb{R}}a_o\left(x,t\right)\varphi\left(x,t\right)f\left(x,t\right)u\left(x,t\right)dxdt-R_0-R_1=\int_{\Omega\times\mathbb{R}}\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}A_{0,P,Q}\left(x_o,\xi_{o3}\right)f\left(x,t\right)\ u\left(x,t\right)dxdt\tag{2.7}$$

where

$$R_1=-\int_{\Omega\times\mathbb{R}}\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}\left[\sum_{n=1}^{2P+1}A_0^n\left(x_o,\xi_{o3}\right)f\left(x,t\right)+\sum_{-2Q\leq n\leq -1}A_0^n\left(x_o,\xi_{o3}\right)f\left(x,t\right)\right]u\left(x,t\right)dxdt\ .$$

We estimate R_1 uniformly with respect to (P,Q) as follows.

$$\begin{array}{ll}R_1&\leq\int_{\Omega\times\mathbb{R}}\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}\sum_{n\in\mathbb{Z}\backslash\{0\}}\left|A_0^n\left(x_o,\xi_{o3}\right)f\left(x,t\right)\right|\ \left|u\left(x,t\right)\right|dxdt\\ &\leq\int_{\Omega\times\mathbb{R}}\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}\frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\ d\xi d\tau\\ &\qquad\sum_{n\in\mathbb{Z}\backslash\{0\}}a\left(x_1-x_{o1},x_2-x_{o2},(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3},t,0\right)\ \left|u\left(x,t\right)\right|dxdt\\ &\leq\frac{1}{(2\pi)^4}\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\ d\xi d\tau\\ &\qquad\int_{\Omega\times\mathbb{R}}\sum_{n\in\mathbb{Z}\backslash\{0\}}a\left(x_1-x_{o1},x_2-x_{o2},(-1)^n x_3+2n\rho-x_{o3},t,0\right)\ \left|u\left(x,t\right)\right|dxdt\ .\end{array}$$

Now, notice that

$$\begin{aligned} & a\left(x_1-x_{o1},x_2-x_{o2},(-1)^n x_3+2n\tfrac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3},t,0\right) \\ &= e^{-\frac{1}{4h}\left[(x_1-x_{o1})^2+(x_2-x_{o2})^2+\left((-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}\right)^2\right]}\ e^{-\frac{t^2}{4}}\ . \end{aligned}$$

The hypothesis saying that $\Upsilon\subset\mathbb{R}^2\backslash\left((-m_1,m_1)\times(-m_2,m_2)\right)\times\mathbb{R}$ implies that for any $x_o\in\overline{\omega_o}$ and $x=(x_1,x_2,x_3)\in\Omega$ such that $(x_1,x_2)\notin[-m_1+r_o/2,m_1-r_o/2]\times[-m_2+r_o/2,m_2-r_o/2]$, we have $(x_1-x_{o1})^2+(x_2-x_{o2})^2\geq(r_o/2)^2$, but such hypothesis also implies that for any $x_o\in\overline{\omega_o}$ and $x=(x_1,x_2,x_3)\in\Omega$ such that $(x_1,x_2)\in[-m_1+r_o/2,m_1-r_o/2]\times[-m_2+r_o/2,m_2-r_o/2]$, we get $x_3\in[-\rho,\rho]$ and therefore $\left|(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}\right|\geq\frac{3}{4}\rho$ for any $n\in\mathbb{Z}\backslash\{0\}$. So, for any $n\in\mathbb{Z}\backslash\{0\}$

$$a\left(x_1-x_{o1},x_2-x_{o2},(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3},t,0\right)\leq e^{-\frac{c}{h}}e^{-\frac{1}{8h}\left((-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}\right)^2}e^{-\frac{t^2}{4}}\ .$$

It follows that

$$\int_{\Omega\times\mathbb{R}}\sum_{n\in\mathbb{Z}\backslash\{0\}}a\left(x_1-x_{o1},x_2-x_{o2},(-1)^n x_3+2n\rho-x_{o3},t,0\right)\ \left|u(x,t)\right|dxdt\leq c\ e^{-\frac{c}{h}}\sqrt{\mathcal{G}(u,0)}$$

Now it remains to compute $\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|\ d\xi d\tau$. We can check that there exists $\gamma>1$ such that

$$\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|\ d\xi d\tau\leq c\left(\frac{\lambda}{h}\right)^\gamma\sqrt{\mathcal{G}(u,0)}\ . \tag{2.8}$$

The proof of (2.8) is given in Appendix A.

We conclude that

$$R_1\leq c\left(\frac{\lambda}{h}\right)^\gamma e^{-\frac{c}{h}}\mathcal{G}(u,0)\ . \tag{2.9}$$

2.5 On the boundary $\{x_3=\pm\rho\}$

Since $a(x_1,x_2,x_3,t,s)=a(x_1,x_2,-x_3,t,s)$, we get the following identity

$$A_s^n(x_o,\xi_{o3})f\left(x_1,x_2,(-1)^n\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)=-A_s^{n+1}(x_o,\xi_{o3})f\left(x_1,x_2,(-1)^n\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)\quad\forall n\in\mathbb{Z}\ . \tag{2.10}$$

Thus,

$$A_s^{2n}(x_o,\xi_{o3})f\left(x_1,x_2,\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)=-A_s^{2n+1}(x_o,\xi_{o3})f\left(x_1,x_2,\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)\quad\forall n\in\mathbb{Z}\ ,$$

so that

$$\begin{aligned} A_{s,P,Q}(x_o,\xi_{o3})f\left(x_1,x_2,\tfrac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right) &= \sum_{n=-Q}^{P}\left[A_s^{2n}(x_o,\xi_{o3})f(x,t)+A_s^{2n+1}(x_o,\xi_{o3})f(x,t)\right] \\ &= 0\quad\forall s\geq0\ . \end{aligned}$$

Also, (2.10) implies

$$A_s^{2n+1}(x_o,\xi_{o3})f\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)=-A_s^{2n+2}(x_o,\xi_{o3})f\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)\quad\forall n\in\mathbb{Z}\ ,$$

therefore

$$\begin{aligned} & A_{s,P,Q}(x_o,\xi_{o3})f\left(x_1,x_2,-\tfrac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right) \\ &= A_s^{-2Q}(x_o,\xi_{o3})f\left(x_1,x_2,-\tfrac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)+A_s^{2P+1}(x_o,\xi_{o3})f\left(x_1,x_2,-\tfrac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)\quad\forall s\geq0\ . \end{aligned}$$

2.6 The key identity

By multiplying (2.6) by $u(x,t)$ and integrating by parts over $\Omega\times\mathbb{R}\times[0,L]$, we have that for all $(x_o,\xi_{o3})\in\overline{\omega_o}\times(2\mathbb{Z}+1)$, for all $(P,Q)\in\mathbb{N}^2$ and all $L>0$,

$$\begin{array}{rl}\int_{\Omega\times\mathbb{R}} A_{0,P,Q}(x_o,\xi_{o3})\, f(x,t)\, u(x,t)\, dxdt & =\int_{\Omega\times\mathbb{R}} A_{L,P,Q}(x_o,\xi_{o3})\, f(x,t)\, u(x,t)\, dxdt \\ & \quad +ih\int_0^L\int_{\partial\Omega\times\mathbb{R}} A_{s,P,Q}(x_o,\xi_{o3})\, f(x,t)\,\partial_\nu u(x,t)\, dxdtds\ .\end{array}$$

Consequently, combining the later equality with (2.7), we have the following key identity

$$\begin{array}{rl} & \int_{\Omega\times\mathbb{R}} a_o(x,t)\,\varphi(x,t)\, f(x,t)\, u(x,t)\, dxdt-R_0-R_1 \\ = & \int_{\Omega\times\mathbb{R}} \sum\limits_{\xi_{o3}\in(2\mathbb{Z}+1)} A_{L,P(\xi_{o3}),Q(\xi_{o3})}(x_o,\xi_{o3})\, f(x,t)\, u(x,t)\, dxdt \\ & +ih\int_0^L\int_{\partial\Omega\times\mathbb{R}} \sum\limits_{\xi_{o3}\in(2\mathbb{Z}+1)} A_{s,P(\xi_{o3}),Q(\xi_{o3})}(x_o,\xi_{o3})\, f(x,t)\,\partial_\nu u(x,t)\, dxdtds\end{array} \tag{2.11}$$

for any $L>0$ and $(P(\xi_{o3}),Q(\xi_{o3}))\in\mathbb{N}^2$. L will be taken large enough in order that the first term in the second member of (2.11) is polynomially small like $\frac{c}{\sqrt{L}}$ and this uniformly with respect to $(\xi_{o3},P(\xi_{o3}),Q(\xi_{o3}))$ by using the dispersion of (2.1). Next, for each $\xi_{o3}\in(2\mathbb{Z}+1)$, $(P(\xi_{o3}),Q(\xi_{o3}))$ will be chosen dependent on (ξ_{o3},L) in order that the second term in the second member of (2.11) is exponentially small with respect to h on the boundary $\Gamma_1\cup\Gamma_2$.

2.7 The internal term

In this subsection, we study the internal term appearing in (2.11)

$$\int_{\Omega\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} A_{L,P,Q}(x_o,\xi_{o3})\, f(x,t)\, u(x,t)\, dxdt\ .$$

First, we have a uniform bound with respect to (P,Q)

$$\begin{array}{rl} & \int_{\Omega\times\mathbb{R}} \sum\limits_{\xi_{o3}\in(2\mathbb{Z}+1)} A_{L,P,Q}(x_o,\xi_{o3})\, f(x,t)\, u(x,t)\, dxdt \\ = & \int_{\Omega\times\mathbb{R}} \sum\limits_{\xi_{o3}\in(2\mathbb{Z}+1)} \left[\sum\limits_{n=-2Q}^{2P+1} A_L^n(x_o,\xi_{o3})\, f(x,t)\right] u(x,t)\, dxdt \\ \leq & \sum\limits_{\xi_{o3}\in(2\mathbb{Z}+1)}\sum\limits_{n\in\mathbb{Z}} \left|\int_{\Omega\times\mathbb{R}} A_L^n(x_o,\xi_{o3})\, f(x,t)\, u(x,t)\, dxdt\right|\ .\end{array} \tag{2.12}$$

Recall that

$$\begin{array}{l} A_L^n(x_o,\xi_{o3})\, f(x,t) \\ =\frac{1}{(iL+1)^{3/2}}\frac{(-1)^n}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda} e^{i(x_1\xi_1+x_2\xi_2+t\tau)}\ e^{i\left[(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho\right]\xi_3}\ e^{-i(\xi^2-\tau^2)hL}\ \widehat{\varphi f}(\xi,\tau) \\ \quad e^{-\frac{1}{4h}\frac{(x_1-x_{o1}-2\xi_1 hL)^2}{iL+1}}\ e^{-\frac{1}{4h}\frac{(x_2-x_{o2}-2\xi_2 hL)^2}{iL+1}}\ e^{-\frac{1}{4h}\frac{\left((-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}-2\xi_3 hL\right)^2}{iL+1}} \\ \quad \left(\frac{1}{\sqrt{-ihL+1}}\ e^{-\frac{1}{4}\frac{(t+2\tau hL)^2}{-ihL+1}}\right) d\xi d\tau \end{array}$$

$$\begin{array}{rl} u(x,t) & =\sum\limits_{j\geq1}\left[\frac{b_j^0}{2}\left(e^{it\sqrt{\mu_j}}+e^{-it\sqrt{\mu_j}}\right)+\frac{b_j^1}{2i\sqrt{\mu_j}}\left(e^{it\sqrt{\mu_j}}-e^{-it\sqrt{\mu_j}}\right)\right]\ \ell_j(x) \\ & =\sum\limits_{j\geq1}\left(\frac{b_j^0}{2}+\frac{b_j^1}{2i\sqrt{\mu_j}}\right)e^{it\sqrt{\mu_j}}\ \ell_j(x)+\sum\limits_{j\geq1}\left(\frac{b_j^0}{2}-\frac{b_j^1}{2i\sqrt{\mu_j}}\right)e^{-it\sqrt{\mu_j}}\ \ell_j(x)\ .\end{array}$$

Therefore, we write

$$
\begin{aligned}
&\int_{\Omega\times\mathbb{R}} A_L^n\left(x_o,\xi_{o3}\right) f\left(x,t\right) u\left(x,t\right) dxdt \\
&= \tfrac{1}{(iL+1)^{3/2}} \tfrac{(-1)^n}{(2\pi)^4} \sum_{j\geq 1} \int_\Omega \ell_j\left(x\right) \\
&\quad \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} e^{i(x_1\xi_1+x_2\xi_2)}\ e^{i\left[(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho\right]\xi_3}\ e^{-i\left(\xi^2-\tau^2\right)hL}\ \widehat{\varphi f}\left(\xi,\tau\right)\ d\xi \\
&\quad\quad e^{-\frac{1}{4h}\frac{(x_1-x_{o1}-2\xi_1 hL)^2}{iL+1}}\ e^{-\frac{1}{4h}\frac{(x_2-x_{o2}-2\xi_2 hL)^2}{iL+1}}\ e^{-\frac{1}{4h}\frac{\left((-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}-2\xi_3 hL\right)^2}{iL+1}}\ dx \\
&\quad\quad \int_{\mathbb{R}} e^{it\tau}\left(\tfrac{1}{\sqrt{-ihL+1}}\ e^{-\frac{1}{4}\frac{(t+2\tau hL)^2}{-ihL+1}}\right)\left[\left(\tfrac{b_j^0}{2}+\tfrac{b_j^1}{2i\sqrt{\mu_j}}\right)e^{it\sqrt{\mu_j}}\right. \\
&\quad\quad\quad\quad \left.+\left(\tfrac{b_j^0}{2}-\tfrac{b_j^1}{2i\sqrt{\mu_j}}\right)e^{-it\sqrt{\mu_j}}\right] dt\ d\tau\ .
\end{aligned}
\tag{2.13}
$$

On another hand,

$$
\begin{aligned}
\int_{\mathbb{R}} e^{i\left(\tau\pm\sqrt{\mu_j}\right)t}\ e^{-\frac{1}{4}\frac{(t+2\tau hL)^2}{-ihL+1}}\ dt &= \int_{\mathbb{R}} e^{i\left(\tau\pm\sqrt{\mu_j}\right)(t-2\tau hL)}\ e^{-\frac{1}{4}\frac{t^2}{-ihL+1}}\ dt \\
&= e^{i\left(\tau\pm\sqrt{\mu_j}\right)(-2\tau hL)} \int_{\mathbb{R}} e^{i\left(\tau\pm\sqrt{\mu_j}\right)t}\ e^{-\frac{1}{4}\frac{t^2}{-ihL+1}}\ dt \\
&= e^{i\left(\tau\pm\sqrt{\mu_j}\right)(-2\tau hL)}\ \widehat{e^{-\frac{1}{4}\frac{t^2}{-ihL+1}}}\left(-\left(\tau\pm\sqrt{\mu_j}\right)\right) \\
&= e^{i\left(\tau\pm\sqrt{\mu_j}\right)(-2\tau hL)}\ 2\sqrt{\pi}\sqrt{-ihL+1}\ e^{-(-ihL+1)\left(\tau\pm\sqrt{\mu_j}\right)^2}
\end{aligned}
\tag{2.14}
$$

where we have used the following formula

$$
\widehat{e^{-\frac{z}{2}t^2}}\left(\tau\right)=\int_{\mathbb{R}} e^{-it\tau}\ e^{-\frac{z}{2}t^2}dt=\frac{\sqrt{2\pi}}{\sqrt{z}}\ e^{-\frac{1}{2z}\tau^2}, \quad \operatorname{Re} z\geq 0 \text{ and } z\neq 0\ .
$$

Consequently, by (2.13)-(2.14),

$$
\begin{aligned}
&\int_{\Omega\times\mathbb{R}} A_L^n\left(x_o,\xi_{o3}\right) f\left(x,t\right) u\left(x,t\right) dxdt \\
&= \tfrac{1}{(iL+1)^{3/2}} \tfrac{(-1)^n}{(2\pi)^4} \sum_{j\geq 1} \int_\Omega \ell_j\left(x\right) \\
&\quad \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} e^{i(x_1\xi_1+x_2\xi_2)}\ e^{i\left[(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho\right]\xi_3}\ e^{-i\left(\xi^2-\tau^2\right)hL}\ \widehat{\varphi f}\left(\xi,\tau\right) \\
&\quad\quad e^{-\frac{1}{4h}\frac{(x_1-x_{o1}-2\xi_1 hL)^2}{iL+1}}\ e^{-\frac{1}{4h}\frac{(x_2-x_{o2}-2\xi_2 hL)^2}{iL+1}}\ e^{-\frac{1}{4h}\frac{\left((-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}-2\xi_3 hL\right)^2}{iL+1}}\ dxd\xi \\
&\quad \left[\left(\tfrac{b_j^0}{2}+\tfrac{b_j^1}{2i\sqrt{\mu_j}}\right)e^{i\left(\tau+\sqrt{\mu_j}\right)(-2\tau hL)}\ 2\sqrt{\pi}e^{-(-ihL+1)\left(\tau+\sqrt{\mu_j}\right)^2}\right. \\
&\quad\quad \left.+\left(\tfrac{b_j^0}{2}-\tfrac{b_j^1}{2i\sqrt{\mu_j}}\right)e^{i\left(\tau-\sqrt{\mu_j}\right)(-2\tau hL)}\ 2\sqrt{\pi}e^{-(-ihL+1)\left(\tau-\sqrt{\mu_j}\right)^2}\right]\ d\tau\ .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left|\int_{\Omega\times\mathbb{R}} A_L^n\left(x_o,\xi_o\right) f\left(x,t\right) u\left(x,t\right) dxdt\right| \\
&\leq \tfrac{1}{\left(\sqrt{L^2+1}\right)^{3/2}} \tfrac{2\sqrt{\pi}}{(2\pi)^4} \sum_{j\geq 1} \int_\Omega \left|\ell_j\left(x\right)\right| \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} \left|\widehat{\varphi f}\left(\xi,\tau\right)\right| \\
&\quad\quad e^{-\frac{1}{4h}\left((-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}-2\xi_3 hL\right)^2\frac{1}{L^2+1}}\ dx \\
&\quad\quad \left(\left|\tfrac{b_j^0}{2}+\tfrac{b_j^1}{2i\sqrt{\mu_j}}\right|+\left|\tfrac{b_j^0}{2}-\tfrac{b_j^1}{2i\sqrt{\mu_j}}\right|\right) d\xi d\tau\ .
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}}\left|\int_{\Omega\times\mathbb{R}} A_L^n\left(x_o,\xi_{o3}\right) f\left(x,t\right) u\left(x,t\right) dxdt\right| \\
&\leq c\tfrac{1}{\left(\sqrt{L^2+1}\right)^{3/2}}\left(4+c\sqrt{h\left(L^2+1\right)}\right)\sum_{j\geq 1}\left(\left|b_j^0\right|+\left|\tfrac{b_j^1}{\sqrt{\mu_j}}\right|\right)\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right| d\xi d\tau
\end{aligned}
\tag{2.15}
$$

by using

$$
\sum_{n\in\mathbb{Z}} e^{-\frac{1}{4h}\left((-1)^n x_3\frac{|\xi_{o3}|}{\xi_{o3}}+2n\rho-x_{o3}\frac{|\xi_{o3}|}{\xi_{o3}}-2\xi_3 hL\frac{|\xi_{o3}|}{\xi_{o3}}\right)^2\frac{1}{L^2+1}}\leq 4+c\sqrt{h\left(L^2+1\right)}\ ,
$$

and $\int_\Omega |\ell_j(x)|\, dx \leq c \|\ell_j\|_{L^2(\Omega)} = c$. On another hand, using Cauchy-Schwartz inequality, we have

$$\begin{array}{ll} \sum_{j\geq 1} \left(\left|b_j^0\right| + \left|\frac{b_j^1}{\sqrt{\mu_j}}\right| \right) & \leq \sqrt{\sum_{j\geq 1} \mu_j^2 \left(\left|b_j^0\right| + \left|\frac{b_j^1}{\sqrt{\mu_j}}\right| \right)^2} \sqrt{\sum_{j\geq 1} \left|\frac{1}{\mu_j}\right|^2} \\ & \leq c\sqrt{\mathcal{G}(\partial_t u, 0)} \end{array} \tag{2.16}$$

because

$$\sum_{j\geq 1} \left|\frac{1}{\mu_j}\right|^2 \leq c \sum_{j\geq 1} \left|\frac{1}{j^{2/3}}\right|^2 < +\infty \ .$$

We conclude from (2.12), (2.15)-(2.16) and (2.8), that for any $(P,Q)\in\mathbb{N}^2$,

$$\begin{array}{l} \int_{\Omega\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} A_{L,P,Q}(x_o,\xi_{o3})\, f(x,t)\, u(x,t)\, dxdt \\ \leq c\frac{1}{1+L\sqrt{L}} \left(1+\sqrt{h}L\right) \int_{\mathbb{R}^3}\int_{|\tau|<\lambda} \left|\widehat{\varphi f}(\xi,\tau)\right| d\xi d\tau \ \sqrt{\mathcal{G}(\partial_t u, 0)} \\ \leq c\frac{1}{\sqrt{L}} \left(\frac{\lambda}{h}\right)^\gamma \sqrt{\mathcal{G}(u,0)}\sqrt{\mathcal{G}(\partial_t u, 0)} \ . \end{array} \tag{2.17}$$

2.8 The boundary term

In this subsection, we study the boundary term appearing in (2.11)

$$ih\int_0^L \int_{\partial\Omega\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} A_{s,P(\xi_{o3}),Q(\xi_{o3})}(x_o,\xi_{o3})\, f(x,t)\, \partial_\nu u(x,t)\, dxdtds \ .$$

Recall that $\partial\Omega = \Gamma_1\cup\Gamma_2\cup\Upsilon$. We begin to estimate

$$ih\int_0^L \int_{(\Gamma_1\cup\Gamma_2)\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} A_{s,P(\xi_{o3}),Q(\xi_{o3})}(x_o,\xi_{o3})\, f(x,t)\, \partial_\nu u(x,t)\, dxdtds \ .$$

First, it holds

$$\begin{array}{rl} & \int_0^L \int_{(\Gamma_1\cup\Gamma_2)\times\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} A_{s,P(\xi_{o3}),Q(\xi_{o3})}(x_o,\xi_{o3})\, f(x,t)\, \partial_\nu u(x,t)\, dxdtds \\ \leq & \int_0^L \int_{-m_1}^{m_1}\int_{-m_2}^{m_2}\int_{\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \left| A_{s,P(\xi_{o3}),Q(\xi_{o3})}(x_o,\xi_{o3})\, f\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)\right| \\ & \qquad \left|\partial_{x_3} u\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)\right| dx_1dx_2dtds \\ & + \int_0^L \int_{-m_1}^{m_1}\int_{-m_2}^{m_2}\int_{\mathbb{R}} \sum_{\xi_{o3}\in(2\mathbb{Z}+1)} \left| A_{s,P(\xi_{o3}),Q(\xi_{o3})}(x_o,\xi_{o3})\, f\left(x_1,x_2,\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)\right| \\ & \qquad \left|\partial_{x_3} u\left(x_1,x_2,\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)\right| dx_1dx_2dtds \ . \end{array} \tag{2.18}$$

Recall that for any $(P,Q)\in\mathbb{N}^2$,

$$A_{s,P,Q}(x_o,\xi_{o3})\, f\left(x_1,x_2,\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right) = 0 \quad \forall s\geq 0 \ , \tag{2.19}$$

$$\begin{array}{l} A_{s,P,Q}(x_o,\xi_{o3})\, f\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right) \\ = A_s^{-2Q}(x_o,\xi_{o3})\, f\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right) + A_s^{2P+1}(x_o,\xi_{o3})\, f\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right) \quad \forall s\geq 0 \ , \end{array}$$

where

$$A_s^{-2Q}\left(x_o,\xi_{o3}\right)f\left(x_1,x_2,-\tfrac{\xi_{o3}}{\left|\xi_{o3}\right|}\rho,t\right)$$
$$\leq \tfrac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\ \left|a\left(0,0,(4Q+1)\tfrac{\xi_{o3}}{\left|\xi_{o3}\right|}\rho+x_{o3}+2\xi_3 hs,t+2\tau hs,s\right)\right|\ d\xi d\tau\ ,$$

$$A_s^{2P+1}\left(x_o,\xi_{o3}\right)f\left(x_1,x_2,-\tfrac{\xi_{o3}}{\left|\xi_{o3}\right|}\rho,t\right)$$
$$\leq \tfrac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\ \left|a\left(0,0,(4P+3)\tfrac{\xi_{o3}}{\left|\xi_{o3}\right|}\rho-x_{o3}-2\xi_3 hs,t+2\tau hs,s\right)\right|\ d\xi d\tau\ .$$

Now, for $L>0$, $\xi_{o3}\in(2\mathbb{Z}+1)$ and $(x_{o3},\xi_3)\in\left[-\frac{\rho}{4},\frac{\rho}{4}\right]\times[\xi_{o3}-1,\xi_{o3}+1]$, we choose $Q\left(\xi_{o3}\right)$ and $P\left(\xi_{o3}\right)$ large enough, for example

$$\left\{\begin{array}{ll} Q\left(\xi_{o3}\right)= & Q=\frac{1}{4\rho}\left(L+1\right)\\ P\left(\xi_{o3}\right)= & P=\frac{1}{4\rho}\left[\left(L+1\right)+2\left(\left|\xi_{o3}\right|+1\right)h_o L\right]\end{array}\right.$$

in order that for any $s\in[0,L]$, $h\in(0,h_o]$ and $(x_{o3},\xi_3)\in\left[-\frac{\rho}{4},\frac{\rho}{4}\right]\times[\xi_{o3}-1,\xi_{o3}+1]$,

$$\left\{\begin{array}{r}\left|a\left(0,0,(4Q+1)\frac{\xi_{o3}}{\left|\xi_{o3}\right|}\rho+x_{o3}+2\xi_3 hs,0,s\right)\right|\leq\frac{1}{\left(\sqrt{s^2+1}\right)^{3/2}}\frac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}\ e^{-\frac{1}{4h}}\\ \left|a\left(0,0,-(4P+3)\frac{\xi_{o3}}{\left|\xi_{o3}\right|}\rho+x_{o3}+2\xi_3 hs,0,s\right)\right|\leq\frac{1}{\left(\sqrt{s^2+1}\right)^{3/2}}\frac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}\ e^{-\frac{1}{4h}}\ .\end{array}\right.$$

Consequently,

$$\begin{array}{l}\displaystyle\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}\left|A_{s,P(\xi_{o3}),Q(\xi_{o3})}\left(x_o,\xi_{o3}\right)f\left(x_1,x_2,-\tfrac{\xi_{o3}}{\left|\xi_{o3}\right|}\rho,t\right)\right|\left|\partial_{x_3}u\left(x_1,x_2,-\tfrac{\xi_{o3}}{\left|\xi_{o3}\right|}\rho,t\right)\right|\\ \leq e^{-\frac{1}{4h}}\ \frac{1}{\left(\sqrt{s^2+1}\right)^{3/2}}\frac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}\ \int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\ e^{-\frac{(t+2\tau hs)^2}{4}\frac{1}{(hs)^2+1}}\ d\xi d\tau\\ \quad\left(\left|\partial_{x_3}u\left(x_1,x_2,-\rho,t\right)\right|+\left|\partial_{x_3}u\left(x_1,x_2,\rho,t\right)\right|\right)\ ,\end{array}$$

and we conclude, using (2.18)-(2.19), a classical trace theorem and (2.8), that

$$\begin{array}{l}\left|ih\int_0^L\int_{(\Gamma_1\cup\Gamma_2)\times\mathbb{R}}\displaystyle\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}A_{s,P(\xi_{o3}),Q(\xi_{o3})}\left(x_o,\xi_{o3}\right)f\left(x,t\right)\partial_\nu u\left(x,t\right)dxdtds\right|\\ \leq e^{-\frac{1}{4h}}\ c\int_0^L\frac{1}{1+s\sqrt{s}}\frac{1}{1+\sqrt{h}\sqrt{s}}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\ e^{-\frac{(t+2\tau hs)^2}{4}\frac{1}{(hs)^2+1}}\ d\xi d\tau\ \left\|\partial_\nu u\left(\cdot,t\right)\right\|_{L^2(\partial\Omega)}dtds\\ \leq e^{-\frac{1}{4h}}\ c\int_0^L\frac{1}{1+s\sqrt{s}}\frac{1}{1+\sqrt{h}\sqrt{s}}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\ e^{-\frac{(t+2\tau hs)^2}{4}\frac{1}{(hs)^2+1}}\ d\xi d\tau\sqrt{\mathcal{G}\left(\partial_t u,t\right)}\ dtds\\ \leq e^{-\frac{1}{4h}}\ c\sqrt{\mathcal{G}\left(\partial_t u,0\right)}\int_0^L\frac{1}{1+s\sqrt{s}}\frac{1}{1+\sqrt{h}\sqrt{s}}\left(\int_{\mathbb{R}}e^{-\frac{t^2}{4}\frac{1}{(hs)^2+1}}dt\right)ds\ \int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\ d\xi d\tau\\ \leq e^{-\frac{1}{4h}}\ L\ c\sqrt{\mathcal{G}\left(\partial_t u,0\right)}\ \left(\frac{\lambda}{h}\right)^\gamma\sqrt{\mathcal{G}\left(u,0\right)}\ .\end{array}\tag{2.20}$$

Now, we study to following boundary term

$$ih\int_0^L\int_{\Upsilon\times\mathbb{R}}\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}A_{s,P(\xi_{o3}),Q(\xi_{o3})}\left(x_o,\xi_{o3}\right)f\left(x,t\right)\partial_\nu u\left(x,t\right)dxdtds\ .$$

First, it holds

$$A_{s,P,Q}\left(x_o,\xi_{o3}\right)f\left(x,t\right)\leq\sum_{n\in\mathbb{Z}}\left|A_s^n\left(x_o,\xi_{o3}\right)f\left(x,t\right)\right|\ .\tag{2.21}$$

For $\xi_{o3}\in(2\mathbb{Z}+1)$, we have

$$\begin{array}{l}A_s^n\left(x_o,\xi_{o3}\right)f\left(x,t\right)\\ \leq\frac{1}{(2\pi)^4}\frac{1}{\left(\sqrt{s^2+1}\right)^{3/2}}\frac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\\ \quad e^{-\frac{1}{4h}\left((-1)^n x_3\frac{\left|\xi_{o3}\right|}{\xi_{o3}}+2n\rho-x_{o3}\frac{\left|\xi_{o3}\right|}{\xi_{o3}}-2\xi_3 hs\frac{\left|\xi_{o3}\right|}{\xi_{o3}}\right)^2\frac{1}{s^2+1}}\ e^{-\frac{(t+2\tau hs)^2}{4}\frac{1}{(hs)^2+1}}\ d\xi d\tau\end{array}$$

Noticing

$$\sum_{n\in\mathbb{Z}} e^{-\frac{1}{4h}\left((-1)^n x_3 \frac{|\xi_{o3}|}{\xi_{o3}}+2n\rho-x_{o3}\frac{|\xi_{o3}|}{\xi_{o3}}-2\xi_3 hs\frac{|\xi_{o3}|}{\xi_{o3}}\right)^2\frac{1}{s^2+1}} \leq 4+c\sqrt{h\left(s^2+1\right)}\ ,$$

we deduce that

$$\begin{array}{l}\sum_{n\in\mathbb{Z}}\left|A_s^n\left(x_o,\xi_{o3}\right)f\left(x,t\right)\right| \\ \leq c\left(1+\sqrt{h}s\right)\frac{1}{1+s\sqrt{s}}\frac{1}{1+\sqrt{h}\sqrt{s}}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|e^{-\frac{(t+2\tau hs)^2}{4}\frac{1}{(hs)^2+1}}\ d\xi d\tau\ .\end{array}\tag{2.22}$$

On another hand, we get

$$\begin{array}{l}\int_{\Upsilon\times\mathbb{R}}e^{-\frac{(t+2\tau hs)^2}{4}\frac{1}{(hs)^2+1}}\ \left|\partial_\nu u\left(x,t\right)\right|dxdt \\ \leq \int_\Upsilon\int_{|t+2\tau hs|\leq\sqrt{\frac{(hs)^2+1}{h}}}\left|\partial_\nu u\left(x,t\right)\right|dxdt+e^{-\frac{1}{8h}}\int_{\Upsilon\times\mathbb{R}}e^{-\frac{(t+2\tau hs)^2}{8}\frac{1}{(hs)^2+1}}\ \left|\partial_\nu u\left(x,t\right)\right|dxdt \\ \leq \int_\Upsilon\int_{|t|\leq|2\tau hs|+\sqrt{h}s+\frac{1}{\sqrt{h}}}\left|\partial_\nu u\left(x,t\right)\right|dxdt+c\left(1+hs\right)e^{-\frac{1}{8h}}\sqrt{\mathcal{G}\left(\partial_t u,0\right)}\end{array}\tag{2.23}$$

by cutting the integral over $t\in\mathbb{R}$ into two parts and using a classical trace theorem. From (2.21), (2.22), (2.23) and (2.8), we conclude that

$$\begin{array}{l}\left|ih\int_0^L\int_{\Upsilon\times\mathbb{R}}\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}A_{s,P,Q}\left(x_o,\xi_{o3}\right)f\left(x,t\right)\partial_\nu u\left(x,t\right)dxdtds\right| \\ \leq h\int_0^L\int_{\Upsilon\times\mathbb{R}}\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}\sum_{n\in\mathbb{Z}}\left|A_s^n\left(x_o,\xi_{o3}\right)f\left(x,t\right)\right|\left|\partial_\nu u\left(x,t\right)\right|dxdtds \\ \leq h\int_0^L c\left(1+\sqrt{h}s\right)\frac{1}{1+s\sqrt{s}}\frac{1}{1+\sqrt{h}\sqrt{s}}\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|d\xi \\ \qquad\left(\int_{\Upsilon\times\mathbb{R}}e^{-\frac{(t+2\tau hs)^2}{4}\frac{1}{(hs)^2+1}}\left|\partial_\nu u\left(x,t\right)\right|dxdt\right)\ d\tau ds \\ \leq h\int_0^L c\left(1+\sqrt{h}s\right)\frac{1}{1+s\sqrt{s}}\frac{1}{1+\sqrt{h}\sqrt{s}}\left(\left(\frac{\lambda}{h}\right)^\gamma\sqrt{\mathcal{G}\left(u,0\right)}\right) \\ \qquad\left(\int_\Upsilon\int_{|t|\leq 2\lambda hL+\sqrt{h}L+\frac{1}{\sqrt{h}}}\left|\partial_\nu u\left(x,t\right)\right|dxdt+c\left(1+hs\right)\ e^{-\frac{1}{8h}}\sqrt{\mathcal{G}\left(\partial_t u,0\right)}\right)ds \\ \leq L\ c\left(\frac{\lambda}{h}\right)^\alpha\ \left(\int_\Upsilon\int_{|t|\leq\alpha\left(\frac{\lambda}{h}\right)^\alpha}\left|\partial_\nu u\left(x,t\right)\right|^2dxdt\right)^{1/2}\sqrt{\mathcal{G}\left(u,0\right)}+L\ c\left(\frac{\lambda}{h}\right)^\gamma\ e^{-\frac{1}{8h}}\sqrt{\mathcal{G}\left(u,0\right)}\sqrt{\mathcal{G}\left(\partial_t u,0\right)}\ .\end{array}\tag{2.24}$$

Finally, (2.20) and (2.24) imply

$$\begin{array}{l}\left|ih\int_0^L\int_{\partial\Omega\times\mathbb{R}}\sum_{\xi_{o3}\in(2\mathbb{Z}+1)}A_{s,P(\xi_{o3}),Q(\xi_{o3})}\left(x_o,\xi_{o3}\right)f\left(x,t\right)\partial_\nu u\left(x,t\right)dxdtds\right| \\ \leq L\ c\left(\frac{\lambda}{h}\right)^\gamma\left(\int_\Upsilon\int_{|t|\leq\gamma\left(\frac{\lambda}{h}\right)^\gamma}\left|\partial_\nu u\left(x,t\right)\right|^2dxdt\right)^{1/2}\sqrt{\mathcal{G}\left(u,0\right)}+L\ c\left(\frac{\lambda}{h}\right)^\gamma e^{-\frac{1}{4h}}\sqrt{\mathcal{G}\left(u,0\right)}\sqrt{\mathcal{G}\left(\partial_t u,0\right)}\ .\end{array}\tag{2.25}$$

2.9 The choice of λ and L

By (2.5), (2.9), (2.11), (2.17) and (2.25), we obtain, when $f=\partial_t^2 u$,

$$\begin{array}{l}\int_{\Omega\times\mathbb{R}}a_o\left(x,t\right)\varphi\left(x,t\right)\partial_t^2 u\left(x,t\right)u\left(x,t\right)dxdt \\ \leq c\frac{1}{\sqrt{\lambda}}\sqrt{\mathcal{G}\left(u,0\right)}\sqrt{\mathcal{G}\left(\partial_t u,0\right)} \\ \quad+c\left(\frac{\lambda}{h}\right)^\gamma e^{-\frac{c}{h}}\mathcal{G}\left(u,0\right) \\ \quad+c\frac{1}{\sqrt{L}}\ \left(\frac{\lambda}{h}\right)^\gamma\sqrt{\mathcal{G}\left(u,0\right)}\sqrt{\mathcal{G}\left(\partial_t u,0\right)} \\ \quad+L\ c\left(\frac{\lambda}{h}\right)^\gamma\left(\int_\Upsilon\int_{|t|\leq\gamma\left(\frac{\lambda}{h}\right)^\gamma}\left|\partial_\nu u\left(x,t\right)\right|^2dxdt\right)^{1/2}\ \sqrt{\mathcal{G}\left(u,0\right)}+L\ c\left(\frac{\lambda}{h}\right)^\gamma e^{-\frac{c}{h}}\sqrt{\mathcal{G}\left(u,0\right)}\sqrt{\mathcal{G}\left(\partial_t u,0\right)}\ .\end{array}\tag{2.26}$$

We choose $\lambda \geq 1$ and $L \geq 1$ be such that $\frac{1}{h\sqrt{h}}\frac{1}{\sqrt{\lambda}} = \sqrt{h}$ and $\frac{1}{h\sqrt{h}}\frac{1}{\sqrt{L}} \ \left(\frac{\lambda}{h}\right)^{\gamma} = \sqrt{h}$ in order that

$$\begin{array}{l} \frac{1}{h\sqrt{h}} \int_{\Omega\times\mathbb{R}} a_o(x,t)\,\varphi(x,t)\,\partial_t^2 u(x,t)\,u(x,t)\,dxdt \\ \leq c\sqrt{h}\sqrt{\mathcal{G}(u,0)}\sqrt{\mathcal{G}(\partial_t u,0)} + c\left(\frac{1}{h}\right)^{\gamma}\left(\int_\Upsilon \int_{|t|\leq\gamma\left(\frac{1}{h}\right)^{\gamma}} |\partial_\nu u(x,t)|^2\,dxdt\right)^{1/2} \sqrt{\mathcal{G}(u,0)}\ . \end{array} \tag{2.27}$$

By replacing f by u and a by $\widetilde{a}$ solution of $\left(i\partial_s + h\left(\Delta - \partial_t^2\right)\right)\widetilde{a}(x,t,s) = 0$ given by

$$\widetilde{a}(x,t,s) = \left(\frac{1}{(is+1)^{3/2}}\ e^{-\frac{1}{4h}\frac{x^2}{is+1}}\right)\left(\frac{\sqrt{2}}{\sqrt{-ihs+2}}\ e^{-\frac{1}{4}\frac{t^2}{-ihs+2}}\right)$$

such that $\widetilde{a}(x-x_o,t,0) = a_o\left(x,t/\sqrt{2}\right)$,we can argue in a similar fashion than above to show that

$$\begin{array}{l} \frac{1}{h\sqrt{h}} \int_{\Omega\times\mathbb{R}} a_o\left(x,t/\sqrt{2}\right)\varphi\left(x,t/\sqrt{2}\right)u(x,t)\,u(x,t)\,dxdt \\ \leq c\sqrt{h}\sqrt{\mathcal{G}(u,0)}\sqrt{\mathcal{G}(\partial_t u,0)} + c\left(\frac{1}{h}\right)^{\gamma}\left(\int_\Upsilon \int_{|t|\leq\gamma\left(\frac{1}{h}\right)^{\gamma}} |\partial_\nu u(x,t)|^2\,dxdt\right)^{1/2} \sqrt{\mathcal{G}(u,0)}\ . \end{array} \tag{2.28}$$

Consequently, from (2.2), (2.27)-(2.28), we obtain that for any $\left\{x_o^i\right\}_{i\in I} \in \overline{\omega_o}$,

$$\begin{array}{l} \frac{1}{h\sqrt{h}} \int_{\Omega\times\mathbb{R}} \chi_{x_o^i}(x)\left|a\left(x-x_o^i,t,0\right)\partial_t u(x,t)\right|^2 dxdt \\ \leq c\sqrt{h}\sqrt{\mathcal{G}(u,0)}\sqrt{\mathcal{G}(\partial_t u,0)} + c\left(\frac{1}{h}\right)^{\gamma}\left(\int_\Upsilon \int_{|t|\leq\gamma\left(\frac{1}{h}\right)^{\gamma}} |\partial_\nu u(x,t)|^2\,dxdt\right)^{1/2} \sqrt{\mathcal{G}(u,0)}\ , \end{array}$$

and finally, by (2.3),

$$\begin{array}{l} \int_{\omega_o\times(0,T)} |\partial_t u(x,t)|^2\,dxdt \\ \leq C_o\sqrt{h}\sqrt{\mathcal{G}(u,0)}\sqrt{\mathcal{G}(\partial_t u,0)} + C_o\left(\frac{1}{h}\right)^{\gamma}\left(\int_\Upsilon \int_{|t|\leq\gamma\left(\frac{1}{h}\right)^{\gamma}} |\partial_\nu u(x,t)|^2\,dxdt\right)^{1/2} \sqrt{\mathcal{G}(u,0)}\ . \end{array}$$

for some $C_o > 0$ independent of u and h.

3 Proof of the main result

The choice $\omega_o = (-m_1+r_o, m_1-r_o)\times(-m_2+r_o, m_2-r_o)\times\left(-\frac{\varrho}{4},\frac{\varrho}{4}\right)$ allows to invoke the geometrical control condition. Indeed, observe that any generalized ray of $\partial_t^2 - \Delta$ parametrized by $t\in[0,T]$, for some $T<+\infty$ meets $\omega_o\cup\omega$. As a result, we have the following observability estimate. There exists $C>0$ such that for any initial data $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$, the solution u of (1.2) satisfies

$$\|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq C\left(\int_\omega\int_0^T |\partial_t u(x,t)|^2\,dxdt + \int_{\omega_o}\int_0^T |\partial_t u(x,t)|^2\,dxdt\right)\ . \tag{3.1}$$

From Theorem 2, the solution u of (1.2) satisfies the following interpolation inequality. There exist $C>0$ and $\gamma>1$ such that for any $h\in(0,h_o]$,

$$\begin{array}{rl} \int_{\omega_o}\int_0^T |\partial_t u(x,t)|^2\,dxdt & \leq C\left(\frac{1}{h}\right)^{\gamma}\left(\int_\Upsilon\int_{|t|\leq\gamma\left(\frac{1}{h}\right)^{\gamma}} |\partial_\nu u(x,t)|^2\,dxdt\right)^{1/2} \|(u_0,u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} \\ & +C\sqrt{h}\ \|(u_0,u_1)\|_{H^2\cap H_0^1(\Omega)\times H_0^1(\Omega)}\|(u_0,u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)}\ . \end{array} \tag{3.2}$$

It is now known [Li] using the multipliers technique that the normal derivative of the solution of the wave equation with Dirichlet boundary condition satisfies the following inequalities. Let $\psi\in C_0^\infty(\Theta)$ be such that $\psi=1$ on Υ, then there is $c>0$ such that for any $T>0$,

$$\begin{array}{rl} \|\partial_\nu u\|_{L^2(\Upsilon\times(-T,T))} & \leq c\,\|\psi u\|_{H^1(\omega\times(-2T,2T))} \\ & \leq c\left(\|\partial_t u\|_{L^2(\omega\times(-3T,3T))} + \|u\|_{L^2(\omega\times(-3T,3T))}\right)\ . \end{array} \tag{3.3}$$

Consequently, from (3.1)-(3.2)-(3.3), a translation in time, we obtain that the solution u of (1.2) satisfies

$$\begin{array}{ll}\left\|(u_0,u_1)\right\|_{H_0^1(\Omega)\times L^2(\Omega)} & \leq C\left(\frac{1}{h}\right)^{\gamma}\left(\int_\omega\int_0^{\gamma\left(\frac{1}{h}\right)^\gamma}\left|\partial_t u\left(x,t\right)\right|^2 dxdt\right)^{1/2} \\ & +C\left(\frac{1}{h}\right)^{\gamma}\left(\int_\omega\int_0^{\gamma\left(\frac{1}{h}\right)^\gamma}\left|u\left(x,t\right)\right|^2 dxdt\right)^{1/2} \\ & +C\sqrt{h}\ \left\|(u_0,u_1)\right\|_{H^2\cap H_0^1(\Omega)\times H_0^1(\Omega)} \ .\end{array} \tag{3.4}$$

As $\partial_t w$, i.e., the derivative in time of w solution of (1.1), can be seen as a solution of the wave with a second member $-\alpha\left(x\right)\partial_t w$, (3.4) implies with a usual decomposition method, knowing $\alpha>0$ on ω and $\mathcal{E}\left(w,0\right)\leq c\mathcal{E}\left(\partial_t w,0\right)$, that there exist $C'>0$ and $\gamma>1$ such that

$$\begin{array}{ll}\mathcal{E}\left(w,0\right)+\mathcal{E}\left(\partial_t w,0\right) & \leq C'\left(\frac{1}{h}\right)^{\gamma}\int_\Omega\int_0^{\gamma\left(\frac{1}{h}\right)^\gamma}\left(\alpha\left(x\right)\left|\partial_t^2 w\left(x,t\right)\right|^2+\alpha\left(x\right)\left|\partial_t w\left(x,t\right)\right|^2\right)dxdt \\ & +C'h\ \mathcal{E}\left(\partial_t^2 w,0\right) \quad \forall h\in\left(0,h_o\right] \ .\end{array}$$

This later estimate is clearly also true for any $h>h_o$ and some constant $C''>0$, thus we can choose

$$h=\frac{1}{\left(C'+C''\right)}\frac{\mathcal{E}\left(w,0\right)+\mathcal{E}\left(\partial_t w,0\right)}{\mathcal{E}\left(\partial_t^2 w,0\right)}$$

in order that there exists $C>0$ such that

$$\frac{\mathcal{E}(w,0)+\mathcal{E}(\partial_t w,0)}{\mathcal{E}(\partial_t^2 w,0)}\leq C\left(\frac{\mathcal{E}\left(\partial_t^2 w,0\right)}{\mathcal{E}(w,0)+\mathcal{E}(\partial_t w,0)}\right)^{\gamma}\int_\Omega\int_0^{C\left(\frac{\mathcal{E}\left(\partial_t^2 w,0\right)}{\mathcal{E}(w,0)+\mathcal{E}(\partial_t w,0)}\right)^\gamma}\frac{\alpha(x)\left|\partial_t^2 w(x,t)\right|^2+\alpha(x)\left|\partial_t w(x,t)\right|^2}{\mathcal{E}(\partial_t^2 w,0)}dxdt \ .$$

By a translation in time, we obtain that

$$\begin{array}{ll}\frac{\mathcal{E}(w,s)+\mathcal{E}(\partial_t w,s)}{\mathcal{E}(\partial_t^2 w,0)} & \leq C\left(\frac{\mathcal{E}\left(\partial_t^2 w,0\right)}{\mathcal{E}(w,s)+\mathcal{E}(\partial_t w,s)}\right)^{\gamma}\int_\Omega\int_s^{s+C\left(\frac{\mathcal{E}\left(\partial_t^2 w,0\right)}{\mathcal{E}(w,s)+\mathcal{E}(\partial_t w,s)}\right)^\gamma}\frac{\alpha(x)\left|\partial_t^2 w(x,t)\right|^2+\alpha(x)\left|\partial_t w(x,t)\right|^2}{\mathcal{E}(\partial_t^2 w,0)}dxdt \\ & \leq C\left(\frac{\mathcal{E}\left(\partial_t^2 w,0\right)}{\mathcal{E}(w,s)+\mathcal{E}(\partial_t w,s)}\right)^{\gamma}\left[\left(\frac{\mathcal{E}(w,s)+\mathcal{E}(\partial_t w,s)}{\mathcal{E}(\partial_t^2 w,0)}\right)-\left(\frac{\mathcal{E}(w,t)+\mathcal{E}(\partial_t w,t)}{\mathcal{E}(\partial_t^2 w,0)}\right)\Big|_{t=s+C\left(\frac{\mathcal{E}\left(\partial_t^2 w,0\right)}{\mathcal{E}(w,s)+\mathcal{E}(\partial_t w,s)}\right)^\gamma}\right] \ .\end{array}$$

Applying Lemma in Appendix B to

$$\mathcal{F}\left(s\right)=\sigma\ \frac{\mathcal{E}\left(w,s\right)+\mathcal{E}\left(\partial_t w,s\right)}{\mathcal{E}\left(\partial_t^2 w,0\right)}$$

where $\sigma>0$ is taken in order that $\mathcal{F}$ is bounded by one, we get that there are $C>0$ and $\delta>0$, such that for any $t>0$ and any initial data $\left(\widetilde{w}_0,\widetilde{w}_1\right)\in H^3\left(\Omega\right)\cap H_0^1\left(\Omega\right)\times H^2\left(\Omega\right)\cap H_0^1\left(\Omega\right)$, the solution $\widetilde{w}$ of

$$\left\{\begin{array}{lll}\partial_t^2\widetilde{w}-\Delta\widetilde{w}+\alpha\left(x\right)\partial_t\widetilde{w} & =0 & \text{in } \Omega\times\mathbb{R} \\ \widetilde{w}=\Delta\widetilde{w} & =0 & \text{on } \partial\Omega\times\mathbb{R} \\ \left(\widetilde{w}\left(\cdot,0\right),\partial_t\widetilde{w}\left(\cdot,0\right)\right) & =\left(\widetilde{w}_0,\widetilde{w}_1\right) & \text{in } \Omega \ ,\end{array}\right. \tag{3.5}$$

satisfies

$$\int_\Omega\left(\left|\partial_t^2\widetilde{w}\left(x,t\right)\right|^2+\left|\nabla\partial_t\widetilde{w}\left(x,t\right)\right|^2\right)dx\leq\frac{C}{t^\delta}\left\|\left(\widetilde{w}_0,\widetilde{w}_1\right)\right\|^2_{H^3(\Omega)\times H^2(\Omega)} \ . \tag{3.6}$$

Now, it is known [Li] how to deduce from (3.6) that there are $C>0$ and $\delta>0$, such that for any $t>0$ and any initial data $\left(w_0,w_1\right)\in H^2\left(\Omega\right)\cap H_0^1\left(\Omega\right)\times H_0^1\left(\Omega\right)$, the solution w of (1.1) satisfies,

$$\int_\Omega\left(\left|\partial_t w\left(x,t\right)\right|^2+\left|\nabla w\left(x,t\right)\right|^2\right)dx\leq\frac{C}{t^\delta}\left\|\left(w_0,w_1\right)\right\|^2_{H^2(\Omega)\times H^1(\Omega)} \ .$$

Indeed, let $\widetilde{w}_0\in H^3\left(\Omega\right)\cap H_0^1\left(\Omega\right)$ be such that $\Delta\widetilde{w}_0=w_1+\alpha w_0\in H_0^1\left(\Omega\right)$. In particular, it comes

$$\left\|\widetilde{w}_0\right\|^2_{H^3(\Omega)}\leq c\left\|w_1+\alpha w_0\right\|^2_{H_0^1(\Omega)} \ .$$

Now, notice that w solves

$$\left\{ \begin{array}{c} \partial_t w(x,t) - w_1(x) - \int_0^t \Delta w(x,\tau)\, d\tau + \alpha(x)(w(x,t) - w_0(x)) = 0 \ , \quad (x,t) \in \Omega \times \mathbb{R}_+ \ , \\ w = 0 \quad \text{on } \partial\Omega \times \mathbb{R}_+ \ , \\ w(\cdot,0) = w_0,\ \partial_t w(\cdot,0) = w_1 \quad \text{in } \Omega \ , \end{array} \right.$$

and as a result, $\widetilde{w}(x,t) = \int_0^t w(x,\tau)\, d\tau + \widetilde{w}_0(x)$ solves (3.5) with $\partial_t \widetilde{w}(\cdot,0) = w_0$ and satisfies (3.6). And we conclude that

$$\int_\Omega \left(|\partial_t w(x,t)|^2 + |\nabla w(x,t)|^2 \right) dx \leq \frac{C}{t^\delta} \left(c \left\| w_1 + \alpha w_0 \right\|^2_{H^1_0(\Omega)} + \| w_0 \|^2_{H^2(\Omega)} \right) .$$

That completes the proof.

4 Appendix A

The goal of this Appendix is to prove, with the notations of the above sections, the two following inequalities,

$$\begin{array}{l} \left| \int_{\Omega\times\mathbb{R}} a_o(x,t) \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau|\geq\lambda} e^{i(x\xi+t\tau)}\ \widehat{\varphi u}(\xi,\tau)\ d\xi d\tau\ u(x,t)\ dxdt \right| \\ + \left| \int_{\Omega\times\mathbb{R}} a_o(x,t) \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau|\geq\lambda} e^{i(x\xi+t\tau)}\ \widehat{\varphi \partial_t^2 u}(\xi,\tau)\ d\xi d\tau\ u(x,t)\ dxdt \right| \\ \leq c\sqrt{\frac{1}{\lambda}} \sqrt{\mathcal{G}(u,0)} \sqrt{\mathcal{G}(\partial_t u,0)} \end{array} \tag{A1}$$

and

$$\int_{\mathbb{R}^3} \int_{|\tau|<\lambda} \left| \widehat{\varphi u}(\xi,\tau) \right|\ d\xi d\tau + \int_{\mathbb{R}^3} \int_{|\tau|<\lambda} \left| \widehat{\varphi \partial_t^2 u}(\xi,\tau) \right|\ d\xi d\tau \leq c \left(\frac{\lambda}{h} \right)^\gamma \sqrt{\mathcal{G}(u,0)} \ . \tag{A2}$$

Proof of (A1). Introduce

$$\begin{array}{rl} R(g) & = \left| \int_{\Omega\times\mathbb{R}} a_o(x,t) \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau|\geq\lambda} e^{i(x\xi+t\tau)}\ \widehat{\varphi g}(\xi,\tau)\ d\xi d\tau\ u(x,t)\ dxdt \right| \\ & = \left| \int_{\Omega\times\mathbb{R}} a_o(x,t)\ \partial_t \left(\frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau|\geq\lambda} \frac{1}{i\tau}\ e^{i(x\xi+t\tau)}\ \widehat{\varphi g}(\xi,\tau)\ d\xi d\tau \right) u(x,t)\, dxdt \right| \\ & = \left| - \int_{\Omega\times\mathbb{R}} \partial_t (a_o u(x,t)) \left(\frac{1}{2\pi} \int_{|\tau|\geq\lambda} \frac{1}{i\tau}\ e^{it\tau} \left[\int_{\mathbb{R}} e^{-i\theta\tau} \varphi g(x,\theta)\ d\theta \right] d\tau \right)\ dxdt \right| \ . \end{array}$$

It follows using Cauchy-Schwartz inequality and Parseval identity that

$$\begin{array}{rl} R(g) & \leq \int_{\Omega\times\mathbb{R}} |\partial_t (a_o u(x,t))| \left(\frac{1}{2\pi} \sqrt{\int_{|\tau|\geq\lambda} \frac{1}{\tau^2} d\tau} \sqrt{\int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{-i\theta\tau} \varphi g(x,\theta)\, d\theta \right|^2 d\tau} \right) dxdt \\ & \leq \int_{\Omega\times\mathbb{R}} |\partial_t (a_o u(x,t))| \left(\frac{1}{2\pi} \sqrt{\int_{|\tau|\geq\lambda} \frac{1}{\tau^2} d\tau} \sqrt{2\pi \int_{\mathbb{R}} |\varphi g(x,\theta)|^2\, d\theta} \right) dxdt \\ & \leq \int_{\Omega\times\mathbb{R}} |\partial_t (a_o u(x,t))| \left(\frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\lambda}} \left\| \varphi g(x,\cdot) \right\|_{L^2(\mathbb{R})} \right) dxdt \\ & \leq \frac{1}{\sqrt{\pi}} \sqrt{\frac{1}{\lambda}}\ \int_{\mathbb{R}} \left\| \partial_t (a_o u)(\cdot,t) \right\|_{L^2(\Omega)} dt \left\| \varphi g(x,\cdot) \right\|_{L^2(\Omega\times\mathbb{R})} \ . \end{array}$$

Since we have the following estimates

$$\begin{array}{rl} \left\| \varphi u(x,\cdot) \right\|_{L^2(\Omega\times\mathbb{R})} + \left\| \varphi \partial_t^2 u(x,\cdot) \right\|_{L^2(\Omega\times\mathbb{R})} & \leq \sqrt{\int_{\mathbb{R}} e^{-\frac{1}{2}t^2} \int_\Omega |u(x,t)|^2\, dxdt} \\ & \quad + \sqrt{\int_{\mathbb{R}} e^{-\frac{1}{2}t^2} \int_\Omega \left| \partial_t^2 u(x,t) \right|^2 dxdt} \\ & \leq \sqrt{\sqrt{2\pi}} \left(c\sqrt{\mathcal{G}(u,0)} + \sqrt{\mathcal{G}(\partial_t u,0)} \right) \end{array}$$

$$\begin{array}{rl} \int_{\mathbb{R}} \left\| \partial_t (a_o u)(\cdot,t) \right\|_{L^2(\Omega)} dt & \leq \int_{\mathbb{R}} \sqrt{\int_\Omega |\partial_t a_o u(x,t)|^2\, dx} dt + \int_{\mathbb{R}} \sqrt{\int_\Omega |a_o \partial_t u(x,t)|^2\, dx} dt \\ & \leq \int_{\mathbb{R}} \frac{|t|}{2} e^{-\frac{1}{4}t^2} \sqrt{\int_\Omega |u(x,t)|^2 dx} dt + \int_{\mathbb{R}} e^{-\frac{1}{4}t^2} \sqrt{\int_\Omega |\partial_t u(x,t)|^2 dx} dt \\ & \leq c\sqrt{\mathcal{G}(u,0)} \ , \end{array}$$

we conclude that

$$R(u)+R\left(\partial_t^2 u\right)\leq c\sqrt{\frac{1}{\lambda}}\sqrt{\mathcal{G}(u,0)}\sqrt{\mathcal{G}(\partial_t u,0)}\ .$$

That completes the proof of (A1).

Proof of (A2). We estimate $\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi g}(\xi,\tau)\right|\, d\xi d\tau$ where g solves $\partial_t^2 g-\Delta g=0$ in $\Omega\times\mathbb{R}$. Recall that $\varphi(\cdot,t)=\chi a_o(\cdot,t)\in C_0^\infty(\Omega)$. So, using Cauchy-Schwartz inequality and Parseval identity,

$$\begin{aligned}
\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi g}(\xi,\tau)\right|\, d\xi d\tau &= \int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\tfrac{1+|\xi|^2}{1+|\xi|^2}\left|\widehat{\varphi g}(\xi,\tau)\right|\, d\xi d\tau\\
&\leq \int_{|\tau|<\lambda}\sqrt{\int_{\mathbb{R}^3}\tfrac{1}{\left(1+|\xi|^2\right)^2}d\xi}\sqrt{\int_{\mathbb{R}^3}\left|\widehat{(1-\Delta)(\varphi g)}(\xi,\tau)\right|^2 d\xi d\tau}\\
&\leq \pi^2\int_{|\tau|<\lambda}\sqrt{\int_{\mathbb{R}^3}\left|\widehat{(1-\Delta)(\varphi g)}(\xi,\tau)\right|^2 d\xi d\tau}\\
&\leq \pi^2\sqrt{\lambda}\sqrt{\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{(1-\Delta)(\varphi g)}(\xi,\tau)\right|^2 d\xi d\tau}\ .
\end{aligned}$$

Observe that

$$\begin{aligned}
\Delta(\varphi g) &= \varphi\Delta g+2\nabla\varphi\nabla g+\Delta\varphi g\\
&= \varphi\partial_t^2 g+2\varphi(x_o,t)\nabla\varphi(x,0)\nabla g+\Delta\varphi g\\
&= \partial_t^2(\varphi g)-2\partial_t(\partial_t\varphi g)+\left(\Delta\varphi+\partial_t^2\varphi\right)g+2\varphi(x_o,t)\nabla\varphi(x,0)\nabla g\ .
\end{aligned}$$

As a result,

$$\begin{aligned}
\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi g}(\xi,\tau)\right|\, d\xi d\tau &\leq c\sqrt{\lambda}\lambda^2\sqrt{\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\sum_{k=0,1,2}\left|\widehat{\partial_t^{2-k}\varphi g}(\xi,\tau)\right|^2 d\xi d\tau}\\
&\quad +c\sqrt{\lambda}\sqrt{\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\Delta\varphi g}(\xi,\tau)\right|^2 d\xi d\tau}\\
&\quad +c\sqrt{\lambda}\sqrt{\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\varphi(x_o,t)\widehat{\nabla\varphi(x,0)\nabla g}(\xi,\tau)\right|^2 d\xi d\tau}\ .
\end{aligned}$$

In particular, when $g=u$ we obtain, using Parseval identity

$$\begin{aligned}
\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi u}(\xi,\tau)\right|\, d\xi d\tau &\leq c\sqrt{\lambda}\lambda^2\sum_{k=0,1,2}\left\|\partial_t^{2-k}\varphi u\right\|_{L^2(\Omega\times\mathbb{R})}+c\sqrt{\lambda}\left\|\Delta\varphi u\right\|_{L^2(\Omega\times\mathbb{R})}\\
&\quad +c\sqrt{\lambda}\left\|\varphi(x_o,t)\nabla\varphi(x,0)\nabla u\right\|_{L^2(\Omega\times\mathbb{R})}\\
&\leq c\left(\sqrt{\lambda}\lambda^2+c\sqrt{\lambda}\tfrac{1}{h}\right)\int_{\mathbb{R}}e^{-ct^2}\left(\int_\Omega|u(x,t)|^2 dx\right)dt+c\sqrt{\lambda}\tfrac{1}{\sqrt{h}}\sqrt{\mathcal{G}(u,0)}\ .
\end{aligned}$$

On another hand, when $g=\partial_t^2 u$ and using the fact that $|\tau|<\lambda$,

$$\begin{aligned}
\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|\, d\xi d\tau &\leq c\sqrt{\lambda}\lambda^3\sqrt{\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\sum_{k=0,1,2,3}\left|\widehat{\partial_t^{3-k}\varphi\partial_t u}(\xi,\tau)\right|^2 d\xi d\tau}\\
&\quad +c\sqrt{\lambda}\lambda\sqrt{\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\sum_{k=0,1}\left|\widehat{\partial_t^{1-k}\Delta\varphi\partial_t u}(\xi,\tau)\right|^2 d\xi d\tau}\\
&\quad +c\sqrt{\lambda}\lambda^2\sqrt{\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\sum_{k=0,1,2}\left|\widehat{\partial_t^{2-k}\varphi(x_o,t)\nabla\varphi(x,0)\nabla u}(\xi,\tau)\right|^2 d\xi d\tau}\\
&\leq c\left(\sqrt{\lambda}\lambda^3+\sqrt{\lambda}\lambda\tfrac{1}{h}+\sqrt{\lambda}\lambda^2\tfrac{1}{\sqrt{h}}\right)\sqrt{\mathcal{G}(u,0)}\ .
\end{aligned}$$

We conclude that there exist $c>0$ and $\gamma>1$ such that for any $h\in(0,h_o]$ and $\lambda\geq 1$,

$$\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi u}(\xi,\tau)\right|\, d\xi d\tau+\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi\partial_t^2 u}(\xi,\tau)\right|\, d\xi d\tau\leq c\left(\frac{\lambda}{h}\right)^\gamma\sqrt{\mathcal{G}(u,0)}\ .$$

That completes the proof of (A2).

5 Appendix B

Lemma.- *Let $\mathcal{F}$ be a continuous positive decreasing real function on* $[0,+\infty)$ *and bounded by one. Suppose that there are four constants* $c_1 > 1$ *and* $c_2, \beta, \gamma > 0$ *such that*

$$\mathcal{F}(s) \leq c_1 \left(\frac{1}{\mathcal{F}(s)}\right)^{\beta} \left(\mathcal{F}(s) - \mathcal{F}\left(\left(\frac{c_2}{\mathcal{F}(s)}\right)^{\gamma} + s\right)\right) \quad \forall s > 0 \ .$$

Then there exist $C > 0$ *and* $\delta > 0$ *such that for any* $t > 0$,

$$\mathcal{F}(t) \leq \frac{C}{t^{\delta}} \ .$$

Proof .- Let $t > 0$. If $\left(\frac{\mathcal{F}(s)}{c_2}\right)^{\gamma} < \frac{1}{t}$ then $\mathcal{F}(s) \leq \frac{c_2}{t^{1/\gamma}}$. If $\frac{1}{t} \leq \left(\frac{\mathcal{F}(s)}{c_2}\right)^{\gamma}$ then $\left(\frac{c_2}{\mathcal{F}(s)}\right)^{\gamma} + s \leq t + s$, thus $(\mathcal{F}(t+s)) \leq \mathcal{F}\left(\left(\frac{c_2}{\mathcal{F}(s)}\right)^{\gamma} + s\right)$ and therefore

$$\mathcal{F}(s) \leq (c_1)^{\frac{1}{\beta+1}} \left\{\mathcal{F}(s) - (\mathcal{F}(t+s))\right\}^{\frac{1}{\beta+1}} \ .$$

Consequently,

$$\mathcal{F}(s) \leq (c_1)^{\frac{1}{\beta+1}} \left\{\mathcal{F}(s) - (\mathcal{F}(t+s))\right\}^{\frac{1}{\beta+1}} + \frac{c_2}{t^{1/\gamma}} \quad \forall s, t > 0 \ .$$

Let

$$\psi_t(s) = \frac{1}{\left(\frac{c_1 t}{s}\right)^{\frac{1}{\beta+1}} + \frac{c_2}{t^{1/\gamma}}} \ .$$

If $\mathcal{F}(s) - (\mathcal{F}(t+s)) \leq \frac{t}{t+s}$, then $\mathcal{F}(s) \leq \left(\frac{c_1 t}{t+s}\right)^{\frac{1}{\beta+1}} + \frac{c_2}{t^{1/\gamma}}$ and thus $\psi_t(t+s)\mathcal{F}(t+s) \leq 1$. If $\frac{t}{t+s} < \mathcal{F}(s) - (\mathcal{F}(t+s))$, then $\frac{t}{t+s}(\mathcal{F}(s)) \leq \frac{t}{t+s} < \mathcal{F}(s) - (\mathcal{F}(t+s))$. Therefore

$$\begin{aligned}
\psi_t(t+s)\mathcal{F}(t+s) \quad &< \tfrac{s}{t+s}\mathcal{F}(s)\psi_t(t+s) = \psi_t(s)\mathcal{F}(s)\left(\frac{\frac{\psi_t(t+s)}{t+s}}{\frac{\psi_t(s)}{s}}\right) \\
&< \psi_t(s)\mathcal{F}(s)
\end{aligned}$$

by using the decreasing property of $\zeta \longmapsto \frac{\psi_t(\zeta)}{\zeta}$. We have proved that for any $s, t > 0$, we have either $\psi_t(t+s)\mathcal{F}(t+s) \leq 1$, or $\psi_t(t+s)\mathcal{F}(t+s) < \psi_t(s)\mathcal{F}(s)$. In particular, we deduce that for any $t > 0$ and $n \in \mathbb{N}\backslash\{0\}$, either

$$\begin{aligned}
&\psi_t((n+1)t)\mathcal{F}((n+1)t) \leq 1 \\
\text{or} \quad &\psi_t((n+1)t)\mathcal{F}((n+1)t) < \psi_t(nt)\mathcal{F}(nt) \ .
\end{aligned}$$

Then inductively, it implies that

$$\psi_t((n+1)t)\mathcal{F}((n+1)t) \leq \max(1, \psi_t(t)\mathcal{F}(t)) = 1 \ .$$

Hence for all $t > 0$ and $n \in \mathbb{N}\backslash\{0\}$,

$$\mathcal{F}((n+1)t) \leq \left(\frac{c_1}{n+1}\right)^{\frac{1}{\beta+1}} + \frac{c_2}{t^{1/\gamma}}.$$

We choose n such that $n+1 \leq t < n+2$ and we obtain that for all $t \geq 2$,

$$\mathcal{F}(t^2) \leq \left(\frac{2c_1}{t}\right)^{\frac{1}{\beta+1}} + \frac{c_2}{t^{1/\gamma}}.$$

The desired result now follows immediately.

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